E. (a) $\langle \hat{A}^2 \rangle \geq 0$ for Hermitian $\hat{A}$

- Let $\hat{A}$ be a Hermitian Operator
- Any state Ψ : $\langle \hat{A}^2 \rangle = \langle \hat{A}\hat{A} \rangle = \int \Psi^* \hat{A}\,\hat{A}\,\Psi\, d\tau$ ①

$$= \int \Psi^* \hat{A}\,(\hat{A}\Psi)\, d\tau \qquad (\text{c.f. } \int f^* \hat{A}\, g\, d\tau)$$

$$= \int (\hat{A}^* \Psi^*)(\hat{A}\Psi)\, d\tau \qquad (\because \hat{A} \text{ is Hermitian})$$

$$= \int (\hat{A}\Psi)^* (\hat{A}\Psi)\, d\tau \quad ② \qquad (\text{c.f. } \int g^* g\, d\tau)$$

$$= \int \underbrace{|\hat{A}\Psi|^2}_{\geq 0}\, d\tau \geq 0$$

- Notation : $\langle f | g \rangle \equiv \int f^* g\, d\tau$

$$\langle \hat{A}^2 \rangle = \underbrace{\langle \Psi | \hat{A}^2 | \Psi \rangle}_{①} = \underbrace{\langle \hat{A}\Psi | \hat{A}\Psi \rangle}_{②} \geq 0$$

(b) Filling in a proof for $\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle\Psi|[\hat{H},\hat{A}]|\Psi\rangle$

$$\frac{d}{dt}\langle\hat{A}\rangle = \int d\tau\, \frac{\partial\Psi^*}{\partial t}\hat{A}\,\Psi + \int d\tau\, \Psi^*\hat{A}\frac{\partial\Psi}{\partial t}$$

[Assuming $\hat{A}$ does <u>not</u> depend on time. Otherwise, there is a $\frac{\partial\hat{A}}{\partial t}$ term.]

TDSE $\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$, thus[†] $\hat{H}^*\Psi^* = -i\hbar\frac{\partial\Psi^*}{\partial t}$

$$\frac{d}{dt}\langle\hat{A}\rangle = \int d\tau\left(\frac{i\hat{H}^*\Psi^*}{\hbar}\right)\hat{A}\,\Psi + \int d\tau\,\Psi^*\hat{A}\left(\frac{-i\hat{H}\Psi}{\hbar}\right)$$

$$= \frac{i}{\hbar}\int d\tau\,(\hat{H}\Psi)^*\hat{A}\,\Psi - \frac{i}{\hbar}\int d\tau\,\Psi^*\hat{A}\hat{H}\,\Psi$$

$$= \frac{i}{\hbar}\int d\tau\,\Psi^*\hat{H}\hat{A}\,\Psi - \frac{i}{\hbar}\int d\tau\,\Psi^*\hat{A}\hat{H}\,\Psi \quad (\because \hat{H} \text{ is Hermitian})$$

$$= \frac{i}{\hbar}\int d\tau\,\Psi^*[\hat{H},\hat{A}]\Psi = \frac{i}{\hbar}\langle\Psi|[\hat{H},\hat{A}]|\Psi\rangle$$

Ehrenfest's Theorem

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† $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)$, $\hat{H}^* = \hat{H}$, but we don't need it here.

F. General Uncertainty Relation

- Two physical quantities : $\hat{A}$ and $\hat{B}$ (both Hermitian)
- Some state Ψ
- Mean and Uncertainty are about measurements

[Recall: 1 million copies, measure A and record result (throw away state), repeat, ..., mean $\langle\hat{A}\rangle$, uncertainty $\langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle$; then do the same for measuring B ...]

$$(\Delta A)^2 = \int \Psi^* (\underbrace{\hat{A}-\langle\hat{A}\rangle}_{\hat{A}'})^2 \Psi \, d\tau \quad ; \quad \underbrace{\langle\hat{A}\rangle}_{\text{real}} = \int \Psi^* \hat{A} \Psi \, d\tau$$

real ($\because$ $\hat{A}$ is Hermitian)

- $\hat{A} - \langle\hat{A}\rangle \equiv \hat{A}'$ is also Hermitian

just a number

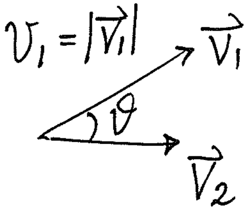
$$(\Delta A)^2 = \int \Psi^* \hat{A}'^2 \Psi \, d\tau = \int |\hat{A}'\Psi|^2 d\tau \qquad (1)$$

Similarly, $(\Delta B)^2 = \int \Psi^* (\underbrace{\hat{B} - \langle \hat{B} \rangle}_{\hat{B}' \text{ (also Hermitian)}})^2 \Psi \, d\tau = \int |\hat{B}' \Psi|^2 d\tau$

$$\therefore \quad (\Delta A)^2 (\Delta B)^2 = \int |\hat{A}' \Psi|^2 d\tau \cdot \int |\hat{B}' \Psi|^2 d\tau \qquad (2)$$

The following step is motivated by

$v_1 = |\vec{V}_1|$



$v_2 = |\vec{V}_2|$

2 vectors

$$\vec{V}_1 \cdot \vec{V}_2 = v_1 v_2 \overbrace{\cos\vartheta}^{\leq 1}$$

$$|\vec{V}_1 \cdot \vec{V}_2|^2 = v_1^2 v_2^2 \cos^2\vartheta$$

$$\leq v_1^2 \quad v_2^2 \qquad (\text{equal if } \vec{V}_1 \parallel \vec{V}_2)$$

$$= |\vec{V}_1|^2 \; |\vec{V}_2|^2$$

$$\therefore \quad |\vec{V}_1|^2 |\vec{V}_2|^2 \geq |\vec{V}_1 \cdot \vec{V}_2|^2 \qquad (3)$$

Recall <u>analogy</u>: Inner product $\left\langle \begin{array}{l} \vec{V}_1 \cdot \vec{V}_2 \quad \text{(vectors)} \\ \int f^* g \, d\tau \quad \text{OR} \quad \langle f | g \rangle \ \text{(functions)} \end{array} \right.$

$$|\vec{V}_1|^2 = \vec{V}_1 \cdot \vec{V}_1 \quad \longleftrightarrow \quad \int f^* f \, d\tau = \int |f|^2 d\tau$$

From $|\vec{V}_1|^2 |\vec{V}_2|^2 \geq |\vec{V}_1 \cdot \vec{V}_2|^2$, we have

$$\boxed{\int |f|^2 d\tau \cdot \int |g|^2 d\tau \geq \left| \int f^* g \, d\tau \right|^2} \quad (4)$$

Schwartz's Inequality

Back to (2): $(\Delta A)^2 (\Delta B)^2 = \int |\hat{A}' \Psi|^2 d\tau \cdot \int |\hat{B}' \Psi|^2 d\tau$

$$\geq \left| \int (\hat{A}' \Psi)^* \hat{B}' \Psi \, d\tau \right|^2 \quad \text{(using (4))}$$

$$= \left| \int \Psi^* \hat{A}' \hat{B}' \Psi \, d\tau \right|^2 \quad (5) \quad (\because \hat{A}' \text{ is Hermitian})$$

$$\hat{A}'\hat{B}' = \frac{1}{2}\hat{A}'\hat{B}' + \frac{1}{2}\hat{A}'\hat{B}' + \frac{1}{2}\hat{B}'\hat{A}' - \frac{1}{2}\hat{B}'\hat{A}'$$

$$= \frac{1}{2}(\hat{A}'\hat{B}' - \hat{B}'\hat{A}') + \frac{1}{2}(\hat{A}'\hat{B}' + \hat{B}'\hat{A}')$$

$$= \frac{1}{i}\underbrace{\frac{i}{2}(\hat{A}'\hat{B}' - \hat{B}'\hat{A}')}_{\text{Hermitian (see Sec. D)}} + \frac{1}{2}\underbrace{(\hat{A}'\hat{B}' + \hat{B}'\hat{A}')}_{\text{Hermitian}} \quad (\because \hat{A}' \,\&\, \hat{B}' \text{ Hermitian})$$

$$\int \Psi^* \hat{A}'\hat{B}'\Psi \, d\tau = \underbrace{\frac{1}{2i}\underbrace{\int \Psi^* i(\hat{A}'\hat{B}' - \hat{B}'\hat{A}')\Psi \, d\tau}_{\text{real}}}_{\text{purely imaginary}} + \frac{1}{2}\underbrace{\int \Psi^* (\hat{A}'\hat{B}' + \hat{B}'\hat{A}')\Psi \, d\tau}_{\text{real}}$$

c.f. $(-ia + b)$

$$\left|\int \Psi^* \hat{A}'\hat{B}'\Psi \, d\tau\right|^2 = \frac{1}{4}\left|\int \Psi^* \underbrace{[\hat{A}', \hat{B}']}_{\text{commutator}}\Psi \, d\tau\right|^2 + \underbrace{\frac{1}{4}\left|\int \Psi^* (\hat{A}'\hat{B}' + \hat{B}'\hat{A}')\Psi \, d\tau\right|^2}_{\geq 0} \quad (6)$$

It follows from (5) that

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \int \Psi^* [\hat{A}', \hat{B}'] \Psi \, d\tau \right|^2 \qquad (7)$$

Finally, $[\hat{A}', \hat{B}'] = [\hat{A} - \langle \hat{A} \rangle, \hat{B} - \langle \hat{B} \rangle] = \underbrace{[\hat{A}, \hat{B}]}_{\text{commutator of } \hat{A} \,\&\, \hat{B}}$

just numbers

$$\therefore \quad (\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \int \Psi^* [\hat{A}, \hat{B}] \Psi \, d\tau \right|^2 = \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2$$

expectation value

So

Key and general result

$$\boxed{(\Delta A)(\Delta B) \geq \frac{1}{2} \left| \int \Psi^* [\hat{A}, \hat{B}] \Psi \, d\tau \right| = \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|} \qquad (8)$$

* This is the <u>generalized Uncertainty Relation</u>

Examples: $(\Delta x)(\Delta p) \geq \frac{1}{2}\left|\langle[\hat{x}, \hat{p}]\rangle\right| = \frac{1}{2}\left|\underbrace{\langle i\hbar\rangle}_{i\hbar\int \Psi^*\Psi d\tau \,=\, i\hbar}\right| = \frac{\hbar}{2}$ Well-known result

$$(\Delta L_y)(\Delta L_z) \geq \frac{1}{2}\left|\langle[\hat{L}_y, \hat{L}_z]\rangle\right| = \frac{1}{2}\left|i\hbar\langle\hat{L}_x\rangle\right| = \frac{\hbar}{2}\langle\hat{L}_x\rangle$$

If $[\hat{A}, \hat{B}] = 0$,

$$(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right| = 0$$

$$\therefore (\Delta A)(\Delta B) \geq \underbrace{0}_{}$$

can be zero (which is the case when Ψ is a simultaneous eigenstate of $\hat{A}$ & $\hat{B}$)

Aside: The Schwartz Inequality

$$\int |f+\lambda g|^2 \, d\tau \geq 0 \quad \text{for } \underline{\text{any}} \text{ choice of } \lambda \text{ (possibly complex)}$$

$$\text{LHS} = \int (f+\lambda g)^* (f+\lambda g)\, d\tau = |\lambda|^2 \int g^* g \, d\tau + \lambda \int f^* g \, d\tau + \lambda^* \int g^* f \, d\tau + \int f^* f \, d\tau$$

Choose
$$\lambda = -\frac{\int g^* f \, d\tau}{\int g^* g \, d\tau} = \frac{-\left(\int g f^* d\tau\right)^*}{\int g^* g \, d\tau}$$

$$\text{LHS} = \text{first two terms cancel} - \frac{\left(\int g f^* d\tau\right)\left(\int g^* f \, d\tau\right)}{\int g^* g \, d\tau} + \int f^* f \, d\tau \geq 0$$

$$\Rightarrow \left(\int |f|^2 d\tau\right)\left(\int |g|^2 d\tau\right) \geq \left|\int f^* g \, d\tau\right|^2$$

which is the Schwartz Inequality